The Many-on-One Stochastic Duel

Moshe Kress
CEMA, P.O. Box 2250, Haifa 31021, Israel

The general many-on-one stochastic duel conditioned on the order in which targets are attacked is investigated, and the state probabilities are derived for the first time. The results are illustrated by an example of a three-on-one stochastic duel with negative exponential interfiring times. Some aspects of the tradeoff between individual firepower and the nominal size of a force are investigated.

INTRODUCTION

The marksman problem (one versus a passive target) and the one-on-one stochastic duel has been treated extensively in the past. Ancker [1] provides an excellent survey of the work done by himself and others on these models.

The problem of the general two-on-one stochastic duel was first considered by Gafarian and Ancker [3], who obtained, among other results, closed expressions for the duel state probabilities and the two sides win probabilities.

The many-on-one stochastic duel was considered by Friedman [2] and Kikuta [4] for the case where all the interfiring times (and hence the time required to kill individual units) have the negative exponential distribution (ned). They have obtained an optimal firing policy for the single blue unit.

In this article we extend the general two-on-one stochastic duel model given in [3] to the general many-on-one case conditioned on the order in which targets are attacked. This generalization also accounts for the relaxation of the homogeneity assumption in [3] that all units on the multiunit's (red) side are equivalent in terms of firepower effectiveness and vulnerability to the B unit. In particular, we derive a closed expression for the time-dependent state probabilities and, consequently, the single blue-unit win probability. As an example we use these results to obtain the state probabilities of the three-on-one stochastic duel where all interfiring times are assumed to be ned random variables.

We conclude this article with a short analysis of the required relative effectiveness of the blue unit and a single red unit in order to maintain a "fair fight" between these two sides. It is shown that if the time to a kill by each one of the n red units is a ned variate with the same mean, and by the blue unit is a gamma distributed (gd) variate, then the ratio of the individual mean red time to a kill to the mean blue time to a kill must be approximately proportional to $n(n + 1)$ in order to secure parity.

Model Assumptions and Notations

Consider a situation where a single blue unit (called B) faces n red units $(R_1, \ldots, R_n)$. All units have an unlimited supply of ammunition. The n red units fire continuously and independently of each other.

Naval Research Logistics, Vol. 34, pp. 713–720 (1987)
 CCC 0028-1441/87/050713-08$04.00

Let X_{R_i} be the continuous random interfiring time of R_i which are iid from round to round and let $G_{R_i}(t)$ and $g_{R_i}(t)$ denote the probability distribution function (pdf) and the density function (df) of X_{R_i}, respectively. The probability that R_i killed B on its nth round fired is $P_{R_i}(n)$. If, for example, R_i has a constant conditional kill probability p on each round fired, then $P_{R_i}(n) = (1 - p)^{n-1}p$. In general, $P_{R_i}(n)$ can represent a process with any decreasing, constant, or increasing conditional probability of a kill on round n. In a similar way, we define X_{B_i} to be the interfiring time of B firing at R_i, and $G_{B_i}(t)$ and $g_{B_i}(t)$ to be its pdf and df, respectively. $P_{B_i}(n)$ is the probability that B killed R_i on its nth round fired at R_i. We assume that once B starts firing at a particular R_i, he should not switch his fire until the current unit has been annihilated.

To simplify the derivation of the results, we consider, as in [3], the interkill time, which is the time from the beginning of engagement until kill, rather than the interfiring time. Thus, let S_i be the time it takes B to kill R_i, and let T_i be the time it takes R_i to kill B. According to our assumption, S_i and T_i, $i = 1, \ldots, n$ are independent random variables. As in the case of a marksman firing at a passive target, their distribution functions and density functions are $F_{B_i}(t)$, $f_{B_i}(t)$, $F_{R_i}(t)$, and $f_{R_i}(t)$, respectively, where, for example,

$$F_{B_i}(t) = \sum_{j=1}^{\infty} P_{B_i}(j) G_{B_i}^{j^*}(t)$$

and j^* denotes $j - 1$ convolutions of G_{B_i} with itself.

Let $W_{l,k}(t)$, $l = 0, \ldots, n$, $k = 0,1$, denote the state where exactly l red units and k blue units are already destroyed by time t, and let $q_{l,k}(t) = P_r\{W_{l,k}(t)\}$ represent the corresponding state probability.

Finally, let $F^c_{B_i}(t)$ and $F^c_{R_i}(t)$, $i = 1, \ldots, n$, denote the complementary distribution functions of S_i and T_i, respectively. It is clear that in this general case of the many-on-one stochastic duel, the probabilities $q_{l,k}(t)$ may depend on the firing policy of B. For different orders by which B fires at the R_is, the $q_{l,k}(t)$ probabilities, and hence the win probabilities for each side, may be different. In the particular (homogeneous) case where T_i, $i = 1, \ldots, n$ are iid and S_i, $i = 1, \ldots, n$ are iid, the probabilities $q_{l,k}(t)$ are, of course, independent of that order.

The probabilities that are derived in the next section are, therefore, related to a particular order by which B picks the R_is denoted by $R_1, R_2, \ldots, R_n$. However, to relate the results to the commonly used Lanchester law, we would have to assume a random choice of targets by B. This is equivalent to selecting each of the $n!$ orders with probability $1/n!$, computing $q_{l,k}(t)$ for each order, and summing these $q_{l,k}(t)$s, each weighted by $1/n!$, to get the final answer.

THE STATE PROBABILITIES

To derive the state probabilities $q_{l,k}(t)$ we consider five cases.

CASE I: $l = 0$, $k = 0$. The event $W_{0,0}(t)$ is equivalent to the event $\{S_1 \geq t, \min(T_1, \ldots, T_n) \geq t\}$. Therefore,

$$q_{0,0}(t) = F^c_{B_1}(t) \prod_{i=1}^{n} F^c_{R_i}(t). \tag{1}$$

CASE II: $l = 0, k = 1$. Here $W_{0,1}(t)$ holds if and only if $\{\min(T_1, \ldots, T_n) \le \min(S_1, t)\}$ holds. Hence

$$q_{0,1}(t) = \int_0^t P_r(\min(T_1, \ldots, T_n) \le y) f_{B_1}(y)\, dy + \int_t^\infty P_r(\min(T_1, \ldots, T_n) \le t) f_{B_1}(y)\, dy \qquad (2)$$

or

$$q_{0,1}(t) = 1 - \int_0^t f_{B_1}(y) \prod_{i=1}^n F^c_{R_i}(y)\, dy - F^c_{B_1}(t) \prod_{i=1}^n F^c_{R_i}(t). \qquad (3)$$

CASE III: $l = 1, \ldots, n-1, k = 0$. In this case $W_{l,0}(t)$ holds if and only if

$$\left\{ \sum_{m=1}^{j} S_m \le M_{j,n}, j = 1, \ldots, l, \sum_{m=1}^{l} S_m \le t \le \sum_{m=1}^{l+1} S_m, M_{l+1,n} \ge t \right\},$$

where

$$M_{ij} = \min(T_i, \ldots, T_j), \qquad i < j.$$

It is easily seen that $W_{l,0}(t)$ is equivalent to

$$\left\{ \sum_{m=1}^{j} S_m \le T_j, j = 1, \ldots, l, \sum_{m=1}^{l} S_m \le M_{l+1,n}, \sum_{m=1}^{l} S_m \le t \le \sum_{m=1}^{l+1} S_m, M_{l+1,n} \ge t \right\}.$$

But $M_{l+1,n} \ge t$ and $\Sigma_{m=1}^{l} S_m \le t$ imply that $M_{l+1,n} \ge \Sigma_{m=1}^{l} S_m$. Also, from the independence of $S_1, \ldots, S_n$, $T_1, \ldots, T_n$ it follows that

$$q_{l,0}(t) = \prod_{i=l+1}^{n} F^c_{R_i}(t) P_r \left(\sum_{m=1}^{j} S_m \le T_j, j = 1, \ldots, l, \sum_{m=1}^{l} S_m \le t \le \sum_{m=1}^{l+1} S_m \right). \qquad (4)$$

Now,

$$P_r \left(\sum_{m=1}^{j} S_m \le T_j, j = 1, \ldots, l, \sum_{m=1}^{l} S_m \le t \le \sum_{m=1}^{l+1} S_m \right)$$
$$= \int_0^t f_{B_1}(y_1) F^c_{R_1}(y_1) P_r \left(y_1 + \sum_{m=2}^{l} S_m \le t_j, j = 2, \ldots, l, y_1 + \sum_{m=2}^{l} S_m \le t \le y_1 + \sum_{m=2}^{l+1} S_m \right) dy_1$$

$$= \int_0^t \int_0^{t-y_1} f_{B_1}(y_1) f_{B_2}(y_2) F^c_{R_1}(y_1) F^c_{R_2}(y_1 + y_2)$$

$$\times P_r\left(y_1 + y_2 \sum_{m=3}^{l} S_m \le T_j, j = 3, \ldots, l,\right.$$

$$\left. y_1 + y_2 + \sum_{m=3}^{l} S_m \le t \le y_1 + y_2 + \sum_{m=3}^{l+1} S_m\right) dy_1\, dy_2 = \cdots$$

$$= \int_0^t \int_0^{t-y_1} \cdots \int_0^{t-\sum_{m=1}^{l-2} y_m} \prod_{j=1}^{l-1} f_{B_j}(y_j) F^c_{R_j}\left(\sum_{m=1}^{j} y_m\right)$$

$$\times P_r\left(\sum_{m=1}^{l-1} y_m + S_l \le t \le \sum_{m=1}^{l-1} y_m + S_l\right.$$

$$\left. + S_{l+1}, \sum_{m=1}^{l-1} y_m + S_l \le T_l\right) dy_1 \ldots dy_{l-1}. \tag{5}$$

We conclude that

$$q_{l,0}(t) = \prod_{i=l+1}^{n} F^c_{R_i}(t) \int_0^t \int_0^{t-y_1} \int_0^{t-\sum_{m=1}^{l-1} y_m} \prod_{j=1}^{l} f_{B_j}(y_j) F^c_{R_j}\left(\sum_{m=1}^{j} y_m\right)$$

$$\times F^c_{B_{l+1}}\left(t - \sum_{m=1}^{l} y_m\right) dy_1 \ldots dy_l. \tag{6}$$

CASE IV: $l = 1, \ldots, n-1$, $k = 1$. It can be shown that $W_{l,1}(t)$ holds if and only if

$$\left\{\sum_{m=1}^{j} S_m \le T_j, j = 1, \ldots, l, \sum_{m=1}^{l} S_m \le M_{l+1,n}, \sum_{m=1}^{l+1} S_m \ge M_{l+1,n}, M_{l+1,n} \le t\right\}.$$

Following the $l - 1$ steps as in Case III we obtain that

$$q_{l,1}(t) = \int_0^t \int_0^{t-y_1} \int_0^{t-\sum_{m=1}^{l-1} y_m} \prod_{j=1}^{l} f_{B_j}(y_j) F^c_{R_j}\left(\sum_{m=1}^{j} y_m\right)$$

$$\times P_r\left[\left(\sum_{m=1}^{l} y_m \le M_{l+1,n} \le \min\left(\sum_{m=1}^{l} y_m + S_{l+1}, t\right)\right)\right]. \tag{7}$$

Thus

$$q_{l,1}(t) = \int_0^t \int_0^{t-y_1} \int_0^{t-\sum_{m=1}^{l-1} y_m} \prod_{j=1}^{l} f_{B_j}(y_j) F^c_{R_j}\left(\sum_{m=1}^{j} y_m\right) \left(\prod_{i=l+1}^{n} F^c_{R_i}\left(\sum_{m=1}^{l} y_m\right)\right.$$

$$- \prod_{i=l+1}^{n} F^c_{R_i}(t) F^c_{B_{l+1}}\left(t - \sum_{m=1}^{l} y_m\right) - \int_0^{t-\sum_{m=1}^{l} y_m} \prod_{i=l+1}^{n} F^c_{R_i}\left(\sum_{m=1}^{l+1} y_m\right)$$

$$\left. \times f_{B_{l+1}}(y_{l+1})\, dy_{l+1}\right) dy_1, \ldots, dy_l. \tag{8}$$

CASE V: $l = n$, $k = 0$. The event $W_{n,0}(t)$ holds if and only if

$$\left\{\sum_{m=1}^{j} S_m \le T_j, j = 1, \ldots, n, \sum_{m=1}^{n} S_m \le t\right\}.$$

Applying similar steps as in case III, we obtain that

$$q_{n,0}(t) = \int_0^t \int_0^{t-y_1} \cdots \int_0^{t-\Sigma_{m=1}^{n-1} y_m} \prod_{i=1}^{n} f_{B_i}(y_i) F^c_{R_i}\left(\sum_{m=1}^{i} y_m\right) dy_1, \ldots, dy_n. \quad (9)$$

Since $q_{n,1}(t) \equiv 0$, it follows that the derivation of all state probabilities for the conditional many-on-one stochastic duel has been completed. The probability that blue wins $P[B]$ is

$$P[B] = q_{n,0}(\infty) = \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{n} f_{B_i}(y_i) F^c_{R_i}\left(\sum_{m=1}^{i} y_m\right) dy_1, \ldots, dy_n, \quad (10)$$

and the probability $P[R]$ that the red force wins is

$$P[R] = \sum_{l=0}^{n-1} q_{l,1}(\infty) = \sum_{l=0}^{n-1} \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^{l} F^c_{R_i}\left(\sum_{m=1}^{i} y_m\right)$$
$$\times \prod_{i=1}^{l+1} f_{B_i}(y_i) \left(\prod_{j=l+1}^{n} F^c_{R_j}\left(\sum_{m=1}^{l} y_m - \prod_{j=l+1}^{n} F^c_{R_j}\left(\sum_{m=1}^{l+1} y_m\right)\right)\right)$$
$$\times\, dy_1, \ldots, dy_{l+1} = 1 - P[B]. \quad (11)$$

EXAMPLE

Let S_i and T_i, $i = 1,2,3$ be ned random variables. That is, $f_{B_i}(t) = \lambda e^{-\lambda t}$ and $f_{R_i} = \mu e^{-\mu t}$, $i = 1,2,3$. This is a well-known stochastic Lanchester problem.

Using the results in (1), (3), (6), (8), and (9) we have

$$q_{0,0}(t) = e^{-(\lambda+3\mu)t}, \quad (12)$$

$$q_{1,0}(t) = \frac{\lambda}{\mu} e^{-(\lambda+2\mu)t}(1 - e^{-\mu t}), \quad (13)$$

$$q_{2,0}(t) = \frac{\lambda}{2\mu^2} e^{-(\lambda+\mu)t}(1 - e^{-\mu t})^2, \quad (14)$$

$$q_{3,0}(t) = \lambda^3 \left[\frac{1}{(\lambda+\mu)(\lambda+2\mu)(\lambda+3\mu)} - \frac{e^{-(\lambda+\mu)t}}{2\mu^2(\lambda+\mu)} + \frac{e^{-(\lambda+2\mu)t}}{\mu^2(\lambda+\mu)} - \frac{e^{-(\lambda+3\mu)t}}{2\mu^2(\lambda+3\mu)}\right], \quad (15)$$

$$q_{0,1}(t) = \frac{3\mu}{\lambda+3\mu}(1 - e^{-(\lambda+3\mu)t}), \quad (16)$$

$$q_{1,1}(t) = 2\lambda \left(\frac{\mu}{(\lambda+2\mu)(\lambda+3\mu)} - \frac{e^{-(\lambda+2\mu)t}}{\lambda+2\mu} + \frac{e^{-(\lambda+3\mu)t}}{\lambda+3\mu}\right), \quad (17)$$

$$q_{2,1}(t) = \lambda^2 \left[\frac{\mu}{(\lambda+\mu)(\lambda+2\mu)(\lambda+3\mu)} - \frac{e^{-(\lambda+\mu)t}}{2\mu(\lambda+\mu)} + \frac{e^{-(\lambda+2\mu)t}}{\mu(\lambda+2\mu)} - \frac{e^{-(\lambda+3\mu)t}}{2\mu(\lambda+3\mu)}\right]. \quad (18)$$

From Eq. (6) and the definition of the exponential pdf and df, it follows that for any value of $n = 1,2, \ldots$ the probability $q_{n,0}(t)$ that the blue side wins the battle before time t is obtained by solving the recursion equation

$$q_{n,0}(t) = \lambda \int_0^t q_{(n-1),0}(t-x)e^{-(\lambda+n\mu)x}\,dx. \tag{19}$$

Also, from (10) and (11) we conclude that

$$P[B] = \frac{\lambda^n}{\prod_{i=1}^{n} (\lambda + i\mu)} \tag{20}$$

and

$$P[R] = \sum_{i=0}^{n-1} \frac{(n-1)\lambda^i\mu}{\prod_{j=0}^{i} (\lambda + (n-j)\mu)}. \tag{21}$$

In this exponential case these probabilities may also be written down directly by examining the problem in an appropriate state space.

RELATIVE EFFECTIVENESS OF THE TWO SIDES

When several red units ($R_1, \ldots, R_n$) face in a duel a single blue (B) unit whose firepower is equally effective as that of each R_i, it is clear the odds that the red force wins the battle are higher than that of the single B unit. If, however, the firepower of B may be made more effective, then the classical question of how quality in combat can compensate for quantity arises.

The firepower effectiveness is measured here in terms of kill rate or the reciprocal of the mean time to a kill. The question is, therefore, how much faster, on the average, must the B unit be than each R_i unit in terms of time to a kill, in order to secure a fair fight where $P[B] = \frac{1}{2}$ holds.

Suppose that each S_i, $i = 1, \ldots, n$ is a gamma(α) distributed random variable with scale parameter λ and shape parameter α. Each T_i, $i = 1, \ldots, n$ is a ned random variable with scale parameter μ. That is,

$$f_{B_i}(y) = \lambda^\alpha y^{\alpha-1}e^{-\lambda y}/\Gamma(\alpha), \qquad i = 1, \ldots, n \tag{23}$$

and

$$f_{R_i}(y) = \mu e^{-\mu y}, \qquad i = 1, \ldots, n. \tag{24}$$

The gamma-distributed interkill time may appear in situations where the kill probability function $P_{B_i}(n)$ has a monotonic increasing hazard function; that is, a situation where the conditional kill probability increases with the number of "no-kill" rounds fired. For example, if

$$P_{B_i}(n) = (1 - 1/\beta)^2(n-1)\beta^n, \qquad n = 1,2, \ldots, \qquad 0 < \beta < 1,$$

then it is easily seen that the hazard function $H_{B_i}(n) = P_{B_i}(n)/\Sigma_{j=n}^{\infty} P_{B_i}(j)$ is monotonic increasing and therefore the firing process is with an increasing conditional kill probability given by $(n-1)(1-\beta)^2/((n-1) - (n-2)\beta)$. If the interfiring times are ned random variables with parameter λ, then the interkill time is a gamma-2-distributed random variable with df

$$f_{B_i}(t) = [\lambda(1 - \beta)]^2 t \cdot e^{-\lambda(1-\beta)t}.$$

Let $0 < C_n < 1$ be a number such that $C_n E(T_i) = E(S_i)$ implies $P[B] = 0.5$, where $E(\cdot)$ denotes the mean of the respective distributions. From (23) and (24) it follows that C_n is such that $\lambda C_n = \alpha\mu$ implies $P[B] = 0.5$.

The C_n parameter is the required effectiveness ratio for parity between the two sides. From (10) it follows that C_n must be such that

$$\left(\prod_{i=1}^{n} (1 + iC_n/\alpha)\right)^{\alpha} = 2 \tag{25}$$

holds. The value of the effectiveness ratio C_n depends on the values of the shape parameter of the S_i distribution function.

It is easily seen that as α gets larger, then

$$\left(\prod_{i=1}^{n} (1 + iC_n/\alpha)\right)^{\alpha} \to e^{n(n+1)C_n/2}, \tag{26}$$

and from (25) and (26) $C_n \simeq (2 \ln 2)/n(n + 1)$ for α large enough.

Figure 1 illustrates the graphs of C_n as a function of the size n of the red force. The graphs correspond to the cases where $\alpha = 1$ (ned), $\alpha = 2$, and $\alpha = 5$, and to the limit value of C_n as $\alpha \to \infty$. Clearly, the convergence of C_n is quite rapid and it gets faster as n increases.

An examination of Fig. 1 shows that for $n \gtrsim 3$ the value of C_n is approximately the same for all shapes (α) of the blue force time-to-a-kill (gamma) distributions.

In classical combat analysis it is common to assume that the ratio between

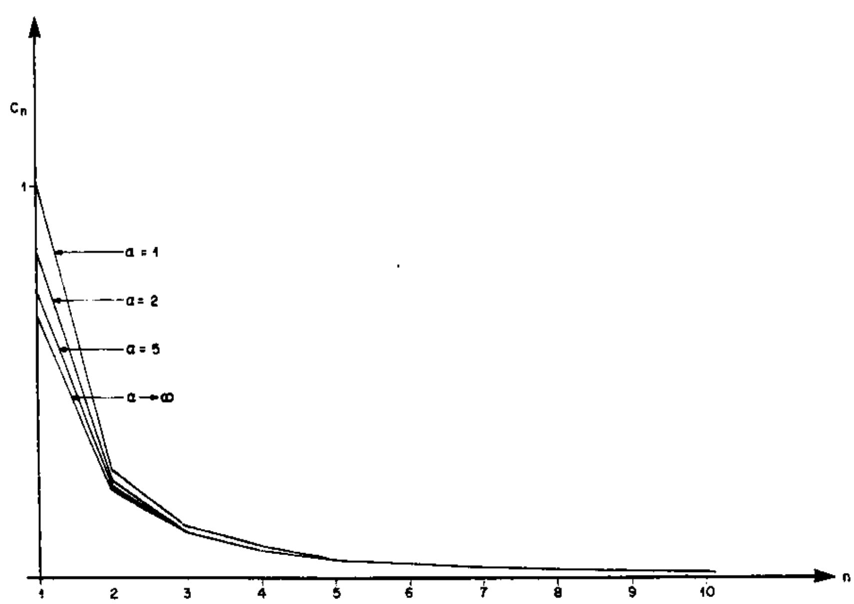

Figure 1. Effectiveness ratio.

the sizes of the attacking and the defending forces is 3 to 1. Ignoring, for that matter, the synergism which is usually present in a multiweapon force, it follows that, for example, a single defending tank must be approximately eight times more effective than an attacking tank in order for the defending force to have equal chances against the three-times-larger attacking force. If, for example, $n = 7$, then the B unit time to a kill must be, on the average, 40 times shorter than each one of the R_i units in order to have $P[B] = 0.5$. The above results are general for all shapes of gamma distributions.

CONCLUSIONS

In this article we have derived, for the first time, the state probabilities and the win probabilities for the general many-on-one duel conditioned on the order in which targets are attacked. These results were illustrated by an example where the interfiring times were ned random variables.

The relation between the effectiveness of the blue unit and a single red unit in terms of time of a kill was investigated. It was shown that if the interkill times are ned random variables on the red side and gamma-distributed random variables on the blue side, then the blue-side effectiveness must be greater than that of a single red unit by a proportion which is on the order of the square of the nominal force ratio. This result is insensitive to the shape parameter $\alpha \geq 1$ of the corresponding gamma distribution.

REFERENCES

[1] Ancker, C. J., Jr., *One-on-One Stochastic Duels*, research monograph, Military Applications Section of the Operations Research Society of America, January 1982.
[2] Friedman, Y., "Optimal Strategy for the One-against-Many Battle," *Operations Research*, **25**(5), 884–888 (1977).
[3] Gafarian, A. V., and Ancker, C. J., Jr., "The Two-on-One Stochastic Duel," *Naval Research Logistics Quarterly*, **31**, 309–324 (1984).
[4] Kikuta, K., "A Note on the One-against-Many Battle," *Operations Research*, **31**(5), 952–956 (1983).

Manuscript received April 9, 1985
Revised manuscript received October 30, 1985
Revised manuscript received June 17, 1986
Revised manuscript received September 16, 1986
Accepted September 18, 1986